# Polynomial Fusion Rings of Logarithmic Minimal Models

Jørgen Rasmussen and Paul A. Pearce

*Department of Mathematics and Statistics, University of Melbourne*
*Parkville, Victoria 3010, Australia*

J.Rasmussen@ms.unimelb.edu.au, P.Pearce@ms.unimelb.edu.au



**Abstract**

We identify quotient polynomial rings isomorphic to the recently found fundamental fusion algebras of logarithmic minimal models.

## 1 Introduction

The fusion algebras of the logarithmic minimal models $\mathcal{LM}(p,p')$ introduced in [1] are discussed in [2, 3]. In these works, it is found that closure of the so-called *fundamental* fusion algebra of $\mathcal{LM}(p,p')$ requires an infinite set of indecomposable representations of rank 1, 2 or 3. The former are so-called Kac representations of which some, but in general only some, are irreducible (highest-weight) representations. It is recalled that the fundamental fusion algebra is so named since it is generated from the two fundamental Kac representations $(2,1)$ and $(1,2)$

$$\big\langle (2,1),(1,2)\big\rangle_{p,p'} \tag{1.1}$$

We let $X$ and $Y$ denote the commuting fusion matrices associated to the representations $(2,1)$ and $(1,2)$, respectively. Since we consider a countably infinite number of representations, $X$ and $Y$ are infinite dimensional. The main objective of the present work is to establish the following proposition where $T_n(x)$ and $U_n(x)$ are Chebyshev polynomials of the first and second kind, respectively, see Appendix A.

**Proposition 1.1** The fundamental fusion algebra of the logarithmic minimal model $\mathcal{LM}(p,p')$ is isomorphic to the polynomial ring generated by $X$ and $Y$ modulo the ideal $\mathcal{I}_{p,p'}(X,Y) = P_{p,p'}(X,Y)\mathbb{C}[X,Y]$ where

$$P_{p,p'}(X,Y) \;=\; \Big(T_p\big(\frac{X}{2}\big)-T_{p'}\big(\frac{Y}{2}\big)\Big)U_{p-1}\big(\frac{X}{2}\big)U_{p'-1}\big(\frac{Y}{2}\big) \tag{1.2}$$

that is,

$$\big\langle (2,1),(1,2)\big\rangle_{p,p'} \;\simeq\; \mathbb{C}[X,Y]/\mathcal{I}_{p,p'}(X,Y) \tag{1.3}$$

It is known [4] that a similar isomorphism exists for every *rational* conformal field theory. The proposition above thus extends this to include the *irrational* logarithmic minimal models as well. We find, though, that the conjectured existence of an associated fusion potential in the case of a rational conformal field theory [4] does not extend to the irrational $\mathcal{LM}(p,p')$, see Appendix B.

## Notation

With $\mathbb{Z}_{n,m} = \mathbb{Z} \cap [n,m]$ denoting the set of integers from $n$ to $m$, both included, we shall be using the following notation: $a \in \mathbb{Z}_{0,p-1}$; $b \in \mathbb{Z}_{0,p'-1}$; $a_0, r_0 \in \mathbb{Z}_{1,p-1}$; $b_0, s_0 \in \mathbb{Z}_{1,p'-1}$.

# 2 Fundamental Fusion Algebra of $\mathcal{LM}(p,p')$

A logarithmic minimal model $\mathcal{LM}(p,p')$ is defined [1] for every coprime pair of positive integers $p < p'$. The model $\mathcal{LM}(p,p')$ has central charge

$$c \;=\; 1 - 6\frac{(p'-p)^2}{pp'} \tag{2.1}$$

and conformal weights

$$\Delta_{r,s} \;=\; \frac{(rp'-sp)^2-(p'-p)^2}{4pp'}, \qquad\qquad r,s \in \mathbb{N} \tag{2.2}$$

## 2.1 Representations

We recall the set of representations $\{\mathcal{R}_{r,s}^{a,b}\}$ appearing in the description of the fundamental fusion algebra of $\mathcal{LM}(p,p')$ [3]. The representation $\mathcal{R}_{r,s}^{a,b}$ is of rank 1 if $a = b = 0$; it is of rank 2 if $a = 0, b \neq 0$ or $a \neq 0, b = 0$; while it is of rank 3 if $a, b \neq 0$. The lower indices $r$ and $s$ are positive integers addressed in the following.

The representations of the form $\mathcal{R}_{r,s}^{0,0}$ are the Kac representations and are also denoted $(r,s)$. In connection with the fundamental fusion algebra, there are three classes of Kac representations: the irreducible, the fully reducible and the reducible yet indecomposable Kac representations

$$\{(r,kp'),(kp,s);\ r \in \mathbb{Z}_{1,p};\ s \in \mathbb{Z}_{1,p'};\ k \in \mathbb{N}\}, \qquad \{(kp,k'p');\ k,k' \in \mathbb{N}+1\}, \qquad \{(r_0,s_0)\} \tag{2.3}$$

here listed in the indicated order.

The higher-rank representations are classified according to their decomposability. For $k,k' \in \mathbb{N}$, $\mathcal{R}_{pk,s_0}^{a_0,0}$ and $\mathcal{R}_{r_0,p'k'}^{0,b_0}$ are indecomposable representations of rank 2; $\mathcal{R}_{pk,p'k'}^{a_0,0}$ and $\mathcal{R}_{pk,p'k'}^{0,b_0}$ are indecomposable representations of rank 2 if $k = 1$ or $k' = 1$ but decomposable representations of rank 2 if $k,k' > 1$; $\mathcal{R}_{pk,p'k'}^{a_0,b_0}$ is an indecomposable representation of rank 3 if $k = 1$ or $k' = 1$ but a decomposable representation of rank 3 if $k,k' > 1$.

The fully reducibility or decomposability of some of these representations is made manifest [3] by

$$\mathcal{R}_{pk,p'k'}^{a,b} \;=\; \bigoplus_{j=|k-k'|+1,\text{ by }2}^{k+k'-1} \mathcal{R}_{pj,p'}^{a,b} \;=\; \bigoplus_{j=|k-k'|+1,\text{ by }2}^{k+k'-1} \mathcal{R}_{p,p'j}^{a,b}, \qquad\quad k,k' \in \mathbb{N} \tag{2.4}$$

Ensuing identifications are

$$\mathcal{R}_{pk,p'k'}^{a,b} \;=\; \mathcal{R}_{pk',p'k}^{a,b} \tag{2.5}$$

with $(kp,p') = (p,kp')$ corresponding to the identification of a pair of irreducible Kac representations of identical conformal weights. The decompositions (2.4) imply that the fundamental fusion algebra of $\mathcal{LM}(p,p')$ can be written in closed form without reference to the fully reducible

Kac representations nor to the decomposable higher-rank representations. Indeed, according to [3], closure of the fundamental fusion algebra requires the inclusion of the representations

$$\big\langle(2,1),(1,2)\big\rangle_{p,p'} \;=\; \big\langle(r_0,s_0),(pk,s_0),(r_0,p'k),(pk,p'),\mathcal{R}_{pk,s_0}^{a_0,0},\mathcal{R}_{pk,p'}^{a_0,0},\mathcal{R}_{r_0,p'k}^{0,b_0},\mathcal{R}_{p,p'k}^{0,b_0},\mathcal{R}_{pk,p'}^{a_0,b_0}\big\rangle_{p,p'} \tag{2.6}$$

where $k\in\mathbb{N}$.

## 2.2 Fusion

The fusion rules of $\mathcal{LM}(p,p')$ are associative, commutative and separate into a horizontal and a vertical part [3]. We indicate this separation with a general but somewhat formal evaluation. Letting $A_{r,s}=\bar{a}_{r,1}\otimes\ a_{1,s}$, $B_{r',s'}=\bar{b}_{r',1}\otimes\ b_{1,s'}$, $\bar{a}_{r,1}\otimes\bar{b}_{r',1}=\bigoplus_{r''}\bar{c}_{r'',1}$ and $a_{1,s}\otimes b_{1,s'}=\bigoplus_{s''}c_{1,s''}$, our fusion prescription yields

$$\begin{aligned}A_{r,s}\otimes B_{r',s'} \;&=\; \Big(\bar{a}_{r,1}\otimes a_{1,s}\Big)\otimes\Big(\bar{b}_{r',1}\otimes b_{1,s'}\Big) \;=\; \Big(\bar{a}_{r,1}\otimes\bar{b}_{r',1}\Big)\otimes\Big(a_{1,s}\otimes b_{1,s'}\Big)\\ &=\; \Big(\bigoplus_{r''}\bar{c}_{r'',1}\Big)\otimes\Big(\bigoplus_{s''}c_{1,s''}\Big) \;=\; \bigoplus_{r'',s''}C_{r'',s''}\end{aligned} \tag{2.7}$$

where $C_{r'',s''}=\bar{c}_{r'',1}\otimes c_{1,s''}$. As illustration, we have

$$\mathcal{R}_{pk,1}^{a,0}\otimes\mathcal{R}_{1,p'k'}^{0,b} \;=\; \mathcal{R}_{pk,p'k'}^{a,b} \tag{2.8}$$

Since the fundamental fusion algebra is built from repeated fusions of the two fundamental representations $(2,1)$ and $(1,2)$, we now list all fusions of one of these fundamental representations with one of the representations (all of which are indecomposable) appearing in (2.6). In the horizontal direction, we have

$$\begin{aligned}(2,1)\otimes(r_0,s_0) \;&=\; (r_0-1,s_0)\oplus(r_0+1,s_0)\\ (2,1)\otimes(pk,s_0) \;&=\; \delta_{p,1}\Big((k-1,s_0)\oplus(k+1,s_0)\Big)\oplus\big(1-\delta_{p,1}\big)\mathcal{R}_{pk,s_0}^{1,0}\\ (2,1)\otimes(r_0,p'k) \;&=\; (r_0-1,p'k)\oplus(r_0+1,p'k)\\ (2,1)\otimes(pk,p') \;&=\; \delta_{p,1}\Big((k-1,p')\oplus(k+1,p')\Big)\oplus\big(1-\delta_{p,1}\big)\mathcal{R}_{pk,p'}^{1,0}\end{aligned} \tag{2.9}$$

and

$$\begin{aligned}
(2,1)\otimes\mathcal{R}_{pk,s_0}^{a_0,0} &= \delta_{p,2}\Big((2k-2,s_0)\oplus 2(2k,s_0)\oplus(2k+2,s_0)\Big)\\
&\oplus\ \big(1-\delta_{p,2}\big)\Big(\big(1+\delta_{a_0,1}\big)\mathcal{R}_{pk,s_0}^{a_0-1,0}\oplus\big(1-\delta_{a_0,p-1}\big)\mathcal{R}_{pk,s_0}^{a_0+1,0}\\
&\qquad\qquad\oplus\ \delta_{a_0,p-1}\big((pk-p,s_0)\oplus(pk+p,s_0)\big)\Big)\\
(2,1)\otimes\mathcal{R}_{pk,p'}^{a_0,0} &= \delta_{p,2}\Big((2k-2,p')\oplus 2(2k,p')\oplus(2k+2,p')\Big)\\
&\oplus\ \big(1-\delta_{p,2}\big)\Big(\big(1+\delta_{a_0,1}\big)\mathcal{R}_{pk,p'}^{a_0-1,0}\oplus\big(1-\delta_{a_0,p-1}\big)\mathcal{R}_{pk,p'}^{a_0+1,0}\\
&\qquad\qquad\oplus\ \delta_{a_0,p-1}\big((pk-p,p')\oplus(pk+p,p')\big)\Big)\\
(2,1)\otimes\mathcal{R}_{r_0,p'k}^{0,b_0} &= \mathcal{R}_{r_0-1,p'k}^{0,b_0}\oplus\mathcal{R}_{r_0+1,p'k}^{0,b_0}\\
(2,1)\otimes\mathcal{R}_{p,p'k}^{0,b_0} &= \delta_{p,1}\Big(\mathcal{R}_{1,p'k-p'}^{0,b_0}\oplus\mathcal{R}_{1,p'k+p'}^{0,b_0}\Big)\oplus\big(1-\delta_{p,1}\big)\mathcal{R}_{pk,p'}^{1,b_0}\\
(2,1)\otimes\mathcal{R}_{pk,p'}^{a_0,b_0} &= \delta_{p,2}\Big(\mathcal{R}_{2k-2,p'}^{0,b_0}\oplus 2\mathcal{R}_{2k,p'}^{0,b_0}\oplus\mathcal{R}_{2k+2,p'}^{0,b_0}\Big)\\
&\oplus\ \big(1-\delta_{p,2}\big)\Big(\big(1+\delta_{a_0,1}\big)\mathcal{R}_{pk,p'}^{a_0-1,b_0}\oplus\big(1-\delta_{a_0,p-1}\big)\mathcal{R}_{pk,p'}^{a_0+1,b_0}\\
&\qquad\qquad\oplus\ \delta_{a_0,p-1}\big(\mathcal{R}_{p,p'k-p'}^{0,b_0}\oplus\mathcal{R}_{p,p'k+p'}^{0,b_0}\big)\Big)
\end{aligned}\tag{2.10}$$

while in the vertical direction we have

$$\begin{aligned}
(1,2)\otimes(r_0,s_0) &= (r_0,s_0-1)\oplus(r_0,s_0+1)\\
(1,2)\otimes(pk,s_0) &= (pk,s_0-1)\oplus(pk,s_0+1)\\
(1,2)\otimes(r_0,p'k) &= \mathcal{R}_{r_0,p'k}^{0,1}\\
(1,2)\otimes(pk,p') &= \mathcal{R}_{p,p'k}^{0,1}
\end{aligned}\tag{2.11}$$

and

$$\begin{aligned}
(1,2)\otimes\mathcal{R}_{pk,s_0}^{a_0,0} &= \mathcal{R}_{pk,s_0-1}^{a_0,0}\oplus\mathcal{R}_{pk,s_0+1}^{a_0,0}\\
(1,2)\otimes\mathcal{R}_{pk,p'}^{a_0,0} &= \mathcal{R}_{pk,p'}^{a_0,1}\\
(1,2)\otimes\mathcal{R}_{r_0,p'k}^{0,b_0} &= \big(1+\delta_{b_0,1}\big)\mathcal{R}_{r_0,p'k}^{0,b_0-1}\oplus\big(1-\delta_{b_0,p'-1}\big)\mathcal{R}_{r_0,p'k}^{0,b_0+1}\\
&\oplus\ \delta_{b_0,p'-1}\big((r_0,p'k-p')\oplus(r_0,p'k+p')\big)\\
(1,2)\otimes\mathcal{R}_{p,p'k}^{0,b_0} &= \delta_{p',2}\big((k-1,2)\oplus 2(k,2)\oplus(k+1,2)\big)\\
&\oplus\ \big(1-\delta_{p',2}\big)\Big(\big(1+\delta_{b_0,1}\big)\mathcal{R}_{p,p'k}^{0,b_0-1}\oplus\big(1-\delta_{b_0,p'-1}\big)\mathcal{R}_{p,p'k}^{0,b_0+1}\\
&\qquad\qquad\oplus\ \delta_{b_0,p'-1}\big((pk-p,p')\oplus(pk+p,p')\big)\Big)\\
(1,2)\otimes\mathcal{R}_{pk,p'}^{a_0,b_0} &= \big(1+\delta_{b_0,1}\big)\mathcal{R}_{pk,p'}^{a_0,b_0-1}\oplus\big(1-\delta_{b_0,p'-1}\big)\mathcal{R}_{pk,p'}^{a_0,b_0+1}\oplus\delta_{b_0,p'-1}\big(\mathcal{R}_{pk-p,p'}^{a_0,0}\oplus\mathcal{R}_{pk+p,p'}^{a_0,0}\big)
\end{aligned}\tag{2.12}$$

Here we have used that $1\leq p<p'$ and introduced the simplifying notation

$$(0,s)\ =\ (r,0)\ =\ \mathcal{R}_{0,s}^{a,b}\ =\ \mathcal{R}_{r,0}^{a,b}\ =\ 0\tag{2.13}$$

Even though we included many details on the fundamental fusion algebras in [2, 3], the lists (2.9) through (2.12) were not presented as explicitly as above. Finally, it is noted that the Kac representation $(1,1)$ is the identity of the fundamental fusion algebra.

# 3 Fundamental Fusion Ring of $\mathcal{LM}(p,p')$

## 3.1 Fusion Matrices and Fusion Rings

The fusion algebra, see [5] for example,

$$\phi_i \otimes \phi_j \;=\; \bigoplus_{k\in\mathcal{J}} \mathcal{N}_{i,j}{}^k \phi_k, \qquad i,j\in\mathcal{J} \tag{3.1}$$

of a *rational* conformal field theory is finite and can be represented by a commutative matrix algebra $\big\langle N_i;\ i\in\mathcal{J}\big\rangle$ where the entries of the square $|\mathcal{J}|\times|\mathcal{J}|$ matrix $N_i$ are

$$\big(N_i\big)_j{}^k \;=\; \mathcal{N}_{i,j}{}^k, \qquad i,j,k\in\mathcal{J} \tag{3.2}$$

and where the fusion product $\otimes$ has been replaced by ordinary matrix multiplication. In [4], Gepner found that every such algebra is isomorphic to a ring of polynomials in a finite set of variables modulo an ideal defined as the vanishing conditions of a finite set of polynomials in these variables. He also conjectured that this ideal of constraints corresponds to the local extrema of a potential, see [6, 7] for further elaborations on this conjecture.

Since the fundamental fusion algebra of the logarithmic minimal model $\mathcal{LM}(p,p')$ contains infinitely many elements, the associated fusion matrices are infinite-dimensional. The corresponding conformal field theory is *irrational* (in this case *logarithmic* [1]) and the results of Gepner [4] do not necessarily apply. We will generally denote these fusion matrices by $N_{(r,s)}$ or $N_{\mathcal{R}_{r,s}^{a,b}}$, cf. (2.6). Associativity of the original commutative fusion algebra ensures that these fusion matrices form a commutative matrix algebra. The fusion matrix associated to the fundamental representation $(2,1)$ or $(1,2)$ is also denoted $X=N_{(2,1)}$ or $Y=N_{(1,2)}$, respectively. As we will argue below, every fusion matrix can be written as a polynomial in $X$ and $Y$ and these polynomials are naturally expressed in terms of Chebyshev polynomials, see Appendix A. With this realization, and in correspondence with a naive extension of the results by Gepner [4], we then identify a quotient polynomial (fusion) ring structure isomorphic to the fundamental fusion algebra of $\mathcal{LM}(p,p')$. There does not, on the other hand, appear to be a fusion potential naturally associated to this fusion ring, see Appendix B. It is emphasized that this is not in violation of Gepner's results since our logarithmic minimal model $\mathcal{LM}(p,p')$ is *irrational*.

As preparation for the derivation of the fusion ring, we now turn our attention to some relations involving Chebyshev polynomials.

## 3.2 Chebyshev Relations

In the following, we consider two possibly non-invertible and possibly non-commuting entities $x$ and $y$ and define the polynomial

$$M_{p,p'}(x,y) \;=\; U_{2p-1}(x)U_{p'-1}(y) - U_{p-1}(x)U_{2p'-1}(y) \tag{3.3}$$

To ease the notation, we will often abbreviate $f(x,y) \equiv g(x,y) \pmod{M_{p,p'}(x,y)}$ simply by $f(x,y) \equiv g(x,y)$.

**Proposition 3.1** For $k \in \mathbb{N}$ and modulo $M_{p,p'}(x,y)$, we have

$$U_{pk-1}(x)U_{p'-1}(y) \;\equiv\; U_{p-1}(x)U_{p'k-1}(y) \tag{3.4}$$

**Proof** This is trivially true for $k = 1, 2$ and we use induction in $k$ to complete the proof. First, though, we prove (3.4) for $k = 3$ in which case

$$\begin{aligned} U_{3p-1}(x)U_{p'-1}(y) &= \Big(2T_p(x)U_{2p-1}(x) - U_{p-1}(x)\Big)U_{p'-1}(y) \\ &\equiv 2T_p(x)U_{p-1}(x)U_{2p'-1}(y) - U_{p-1}(x)U_{p'-1}(y) \\ &= U_{2p-1}(x)U_{p'-1}(y)2T_{p'}(y) - U_{p-1}(x)U_{p'-1}(y) \\ &\equiv U_{p-1}(x)\Big(U_{2p'-1}(y)2T_{p'}(y) - U_{p'-1}(y)\Big) \\ &= U_{p-1}(x)U_{3p'-1}(y) \end{aligned} \tag{3.5}$$

where the three equalities all follow from (A.10). The two equivalences are both immediate consequences of the definition of $M_{p,p'}(x,y)$ in (3.3). To establish the general induction step for $k \geq 3$, we consider

$$\begin{aligned} U_{(k+1)p-1}(x)U_{p'-1}(y) &= \Big(2T_p(x)U_{kp-1}(x) - U_{(k-1)p-1}(x)\Big)U_{p'-1}(y) \\ &\equiv 2T_p(x)U_{p-1}(x)U_{kp'-1}(y) - U_{(k-1)p-1}(x)U_{p'-1}(y) \\ &= 2T_p(x)U_{p-1}(x)\Big(2T_{(k-1)p'}(y)U_{p'-1}(y) + U_{(k-2)p'-1}(y)\Big) \\ &\quad -U_{(k-1)p-1}(x)U_{p'-1}(y) \\ &\equiv U_{p-1}(x)U_{2p'-1}(y)2T_{(k-1)p'}(y) + 2T_p(x)U_{(k-2)p-1}(x)U_{p'-1}(y) \\ &\quad -U_{(k-1)p-1}(x)U_{p'-1}(y) \\ &= U_{p-1}(x)\Big(U_{(k+1)p'-1}(y) - U_{(k-3)p'-1}(y)\Big) \\ &\quad +\Big(U_{(k-1)p-1}(x) + U_{(k-3)p-1}(x)\Big)U_{p'-1}(y) - U_{(k-1)p-1}(x)U_{p'-1}(y) \\ &\equiv U_{p-1}(x)U_{(k+1)p'-1}(y) \end{aligned} \tag{3.6}$$

where, again, all three equalities follow from (A.10), while the three equivalences follow by induction assumption with the second equivalence also relying on (A.10). □

**Proposition 3.2** For $k, k' \in \mathbb{N}$ and modulo $M_{p,p'}(x,y)$, we have

$$U_{pk-1}(x)U_{p'k'-1}(y) \;\equiv\; \sum_{j=|k-k'|+1,\text{ by }2}^{k+k'-1} U_{pj-1}(x)U_{p'-1}(y) \;\equiv\; \sum_{j=|k-k'|+1,\text{ by }2}^{k+k'-1} U_{p-1}(x)U_{p'j-1}(y) \tag{3.7}$$

**Proof** To prove the first equivalence, we initially assume that $k \leq k'$. For $k = 2n+1$ odd and modulo $M_{p,p'}(x,y)$, we then have

$$\begin{aligned} U_{p(2n+1)-1}(x)U_{p'k'-1}(y) &= \Big(1 + 2\sum_{j=1}^{n} T_{2jp}(x)\Big) U_{p-1}(x)U_{p'k'-1}(y) \\ &\equiv \Big(1 + 2\sum_{j=1}^{n} T_{2jp}(x)\Big) U_{pk'-1}(x)U_{p'-1}(y) \\ &= \Big(U_{pk'-1}(x) + \sum_{j=1}^{n} \big(U_{(k'-2j)p-1}(x) + U_{(k'+2j)p-1}(x)\big)\Big) U_{p'-1}(y) \end{aligned} \quad (3.8)$$

which is readily seen to equal the first sum expression of (3.7). The first and second equality of (3.8) follow from (A.11) and (A.10), respectively. For $k = 2n$ even and once again employing (A.11) and (A.10), we likewise have

$$\begin{aligned} U_{2np-1}(x)U_{p'k'-1}(y) &= 2\sum_{j=1}^{n} T_{(2j-1)p}(x)U_{p-1}(x)U_{k'p'-1}(y) \\ &\equiv 2\sum_{j=1}^{n} T_{(2j-1)p}(x)U_{k'p-1}(x)U_{p'-1}(y) \\ &= \sum_{j=1}^{n} \big(U_{(k'+1-2j)p-1}(x) + U_{(k'-1+2j)p-1}(x)\big) U_{p'-1}(y) \end{aligned} \quad (3.9)$$

which is also readily seen to equal the first sum expression of (3.7). The first equivalence of (3.7) for $k > k'$ follows similarly. The second equivalence of (3.7) is a direct consequence of Proposition 3.1. □

**Corollary 3.3** For $k, k' \in \mathbb{N}$ and modulo $M_{p,p'}(x,y)$, we have

$$U_{pk-1}(x)U_{p'k'-1}(y) \equiv U_{pk'-1}(x)U_{p'k-1}(y) \quad (3.10)$$

**Proof** This follows from Proposition 3.2 since the sum expressions of (3.7) are symmetric in $k$ and $k'$. □

## 3.3 Determination of Fusion Matrices and Ring Structure

We now show that the generators of the fundamental fusion algebra (2.6) can be expressed as polynomials in the fusion matrices of the fundamental representations

$$X = N_{(2,1)}, \qquad Y = N_{(1,2)} \quad (3.11)$$

**Proposition 3.4** Modulo the polynomial $P_{p,p'}(X,Y)$ defined in (1.2), the matrices

$$\begin{aligned} N_{(r_0,s_0)}(X,Y) &= U_{r_0-1}\big(\frac{X}{2}\big)U_{s_0-1}\big(\frac{Y}{2}\big) \\ N_{(pk,s_0)}(X,Y) &= U_{pk-1}\big(\frac{X}{2}\big)U_{s_0-1}\big(\frac{Y}{2}\big) \\ N_{(r_0,p'k)}(X,Y) &= U_{r_0-1}\big(\frac{X}{2}\big)U_{p'k-1}\big(\frac{Y}{2}\big) \\ N_{(pk,p')}(X,Y) &= U_{pk-1}\big(\frac{X}{2}\big)U_{p'-1}\big(\frac{Y}{2}\big) \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} N_{\mathcal{R}_{pk,s_0}^{a_0,0}}(X,Y) &= 2T_{a_0}\big(\frac{X}{2}\big)N_{(pk,s_0)}(X,Y) \\ N_{\mathcal{R}_{pk,p'}^{a_0,0}}(X,Y) &= 2T_{a_0}\big(\frac{X}{2}\big)N_{(pk,p')}(X,Y) \\ N_{\mathcal{R}_{r_0,p'k}^{0,b_0}}(X,Y) &= 2N_{(r_0,p'k)}(X,Y)T_{b_0}\big(\frac{Y}{2}\big) \\ N_{\mathcal{R}_{p,p'k}^{0,b_0}}(X,Y) &= 2N_{(p,p'k)}(X,Y)T_{b_0}\big(\frac{Y}{2}\big) \\ N_{\mathcal{R}_{pk,p'}^{a_0,b_0}}(X,Y) &= 4T_{a_0}\big(\frac{X}{2}\big)N_{(pk,p')}(X,Y)T_{b_0}\big(\frac{Y}{2}\big) \end{aligned} \tag{3.13}$$

satisfy the fusion rules (2.9) through (2.12) with the fusion product $\otimes$ and direct summation $\oplus$ replaced by matrix multiplication and addition, respectively. Since every participating representation can be written in the form $\mathcal{R}_{r,s}^{a,b}$, the associated fusion matrix thus reads

$$N_{\mathcal{R}_{r,s}^{a,b}}(X,Y) = \big(2-\delta_{a,0}\big)T_a\big(\frac{X}{2}\big)U_{r-1}\big(\frac{X}{2}\big)\big(2-\delta_{b,0}\big)T_b\big(\frac{Y}{2}\big)U_{s-1}\big(\frac{Y}{2}\big) \tag{3.14}$$

**Proof** There are 18 fusion rules to establish. The first one appears in (2.9) and reads

$$\begin{aligned} (2,1)\otimes(r_0,s_0) &\leftrightarrow XU_{r_0-1}\big(\frac{X}{2}\big)U_{s_0-1}\big(\frac{Y}{2}\big) = \Big(U_{r_0-2}\big(\frac{X}{2}\big)+U_{r_0}\big(\frac{X}{2}\big)\Big)U_{s_0-1}\big(\frac{Y}{2}\big) \\ &\leftrightarrow (r_0-1,s_0)\oplus(r_0+1,s_0) \end{aligned} \tag{3.15}$$

More generally, the task is to decompose the products

$$(2,1)\otimes\mathcal{R}_{r,s}^{a,b} \leftrightarrow X\big(2-\delta_{a,0}\big)T_a\big(\frac{X}{2}\big)U_{r-1}\big(\frac{X}{2}\big)\big(2-\delta_{b,0}\big)T_b\big(\frac{Y}{2}\big)U_{s-1}\big(\frac{Y}{2}\big) \tag{3.16}$$

and

$$(1,2)\otimes\mathcal{R}_{r,s}^{a,b} \leftrightarrow \big(2-\delta_{a,0}\big)T_a\big(\frac{X}{2}\big)U_{r-1}\big(\frac{X}{2}\big)\big(2-\delta_{b,0}\big)YT_b\big(\frac{Y}{2}\big)U_{s-1}\big(\frac{Y}{2}\big) \tag{3.17}$$

in terms of the polynomials (3.12) and (3.13) thereby demonstrating that the fusion rules (2.9) through (2.12) are indeed satisfied. To this end, it is noted that

$$2P_{p,p'}(X,Y) = M_{p,p'}\big(\frac{X}{2},\frac{Y}{2}\big) \tag{3.18}$$

(where $M_{p,p'}(x,y)$ is defined in (3.3)) thus permitting us to draw on Proposition 3.1 and Proposition 3.2. Establishing the remaining 17 fusion rules is now straightforward so we only include one of them as illustration, namely the rule associated to the fusion product

$$(1,2)\otimes\mathcal{R}_{p,p'k}^{0,b_0} \;\leftrightarrow\; U_{p-1}\big(\frac{X}{2}\big)2YT_{b_0}\big(\frac{Y}{2}\big)U_{p'k-1}\big(\frac{Y}{2}\big) \tag{3.19}$$

For $p'=2$, in which case $p=1$ and $b_0=1$, the right side reads

$$\begin{aligned}\Big(2T_1\big(\frac{Y}{2}\big)\Big)^2U_{2k-1}\big(\frac{Y}{2}\big) \;&=\; U_{2k-3}\big(\frac{Y}{2}\big)+2U_{2k-1}\big(\frac{Y}{2}\big)+U_{2k+1}\big(\frac{Y}{2}\big)\\ &\equiv\; U_{k-2}\big(\frac{X}{2}\big)U_1\big(\frac{Y}{2}\big)+2U_{k-1}\big(\frac{X}{2}\big)U_1\big(\frac{Y}{2}\big)+U_k\big(\frac{X}{2}\big)U_1\big(\frac{Y}{2}\big)\\ &\leftrightarrow\; (k-1,2)\oplus 2(k,2)\oplus(k+1,2)\end{aligned} \tag{3.20}$$

where the equivalence is modulo $P_{1,2}(X,Y)$. For $p'>2$, the right side of (3.19) reads

$$\begin{aligned}U_{p-1}\big(\frac{X}{2}\big)2YT_{b_0}\big(\frac{Y}{2}\big)U_{p'k-1}\big(\frac{Y}{2}\big) \;=\; U_{p-1}\big(\frac{X}{2}\big)\Big(&U_{p'k-b_0-2}\big(\frac{Y}{2}\big)+U_{p'k-b_0}\big(\frac{Y}{2}\big)\\ &+U_{p'k+b_0-2}\big(\frac{Y}{2}\big)+U_{p'k+b_0}\big(\frac{Y}{2}\big)\Big)\end{aligned} \tag{3.21}$$

For $b_0=1$, the right side of this equals

$$U_{p-1}\big(\frac{X}{2}\big)\Big(U_{p'k-3}\big(\frac{Y}{2}\big)+2U_{p'k-1}\big(\frac{Y}{2}\big)+U_{p'k+1}\big(\frac{Y}{2}\big)\Big) \;\leftrightarrow\; 2(p,p'k)\oplus\mathcal{R}_{p,p'k}^{0,2} \;=\; 2\mathcal{R}_{p,p'k}^{0,0}\oplus\mathcal{R}_{p,p'k}^{0,2} \tag{3.22}$$

while for $1<b_0<p'-1$, it equals

$$U_{p-1}\big(\frac{X}{2}\big)\Big(U_{p'k-b_0-2}\big(\frac{Y}{2}\big)+U_{p'k-b_0}\big(\frac{Y}{2}\big)+U_{p'k+b_0-2}\big(\frac{Y}{2}\big)+U_{p'k+b_0}\big(\frac{Y}{2}\big)\Big) \;\leftrightarrow\; \mathcal{R}_{p,p'k}^{0,b_0-1}\oplus\mathcal{R}_{p,p'k}^{0,b_0+1} \tag{3.23}$$

whereas it equals

$$\begin{aligned}&U_{p-1}\big(\frac{X}{2}\big)\Big(U_{p'(k-1)-1}\big(\frac{Y}{2}\big)+U_{p'k-(p'-2)-1}\big(\frac{Y}{2}\big)+U_{p'k+(p'-2)-1}\big(\frac{Y}{2}\big)+U_{p'(k+1)-1}\big(\frac{Y}{2}\big)\Big)\\ \equiv\;& U_{p-1}\big(\frac{X}{2}\big)\Big(U_{p'k-(p'-2)-1}\big(\frac{Y}{2}\big)+U_{p'k+(p'-2)-1}\big(\frac{Y}{2}\big)\Big)\\ &+\Big(U_{p(k-1)-1}\big(\frac{X}{2}\big)+U_{p(k+1)-1}\big(\frac{X}{2}\big)\Big)U_{p'-1}\big(\frac{Y}{2}\big)\\ \leftrightarrow\;& \mathcal{R}_{p,p'k}^{0,p'-2}\oplus(p(k-1),p')\oplus(p(k+1),p')\end{aligned} \tag{3.24}$$

for $b_0=p'-1$ where the equivalence is modulo $P_{1,2}(X,Y)$. This completes the proof of the fourth fusion rule of (2.12). □

**Proposition 3.5** The matrices defined by (3.12) and (3.13) in Proposition 3.4 satisfy the fusion prescription outlined in (2.7) with the fusion product $\otimes$ and direct summation $\oplus$ replaced by matrix multiplication and addition, respectively.

**Proof** In analogy with (2.7) and using (3.14), we have

$$
\begin{aligned}
\mathcal{R}_{r,s}^{a,b}\otimes\mathcal{R}_{r',s'}^{a',b'} \;&\leftrightarrow\; \Big\{\big(2-\delta_{a,0}\big)T_a\big(\tfrac{X}{2}\big)U_{r-1}\big(\tfrac{X}{2}\big)\big(2-\delta_{b,0}\big)T_b\big(\tfrac{Y}{2}\big)U_{s-1}\big(\tfrac{Y}{2}\big)\Big\}\\
&\times\; \Big\{\big(2-\delta_{a',0}\big)T_{a'}\big(\tfrac{X}{2}\big)U_{r'-1}\big(\tfrac{X}{2}\big)\big(2-\delta_{b',0}\big)T_{b'}\big(\tfrac{Y}{2}\big)U_{s'-1}\big(\tfrac{Y}{2}\big)\Big\}\\
&=\; \Big\{\big(2-\delta_{a,0}\big)T_a\big(\tfrac{X}{2}\big)U_{r-1}\big(\tfrac{X}{2}\big)\big(2-\delta_{a',0}\big)T_{a'}\big(\tfrac{X}{2}\big)U_{r'-1}\big(\tfrac{X}{2}\big)\Big\}\\
&\times\; \Big\{\big(2-\delta_{b,0}\big)T_b\big(\tfrac{Y}{2}\big)U_{s-1}\big(\tfrac{Y}{2}\big)\big(2-\delta_{b',0}\big)T_{b'}\big(\tfrac{Y}{2}\big)U_{s'-1}\big(\tfrac{Y}{2}\big)\Big\}\\
&=\; \Big\{\sum_{r'',a''}\big(2-\delta_{a'',0}\big)T_{a''}\big(\tfrac{X}{2}\big)U_{r''-1}\big(\tfrac{X}{2}\big)\Big\}\Big\{\sum_{s'',b''}\big(2-\delta_{b'',0}\big)T_{b''}\big(\tfrac{Y}{2}\big)U_{s''-1}\big(\tfrac{Y}{2}\big)\Big\}\\
&\leftrightarrow\; \bigoplus_{r'',s'',a'',b''}\mathcal{R}_{r'',s''}^{a'',b''}
\end{aligned}
\tag{3.25}
$$

□

In terms of the polynomials (3.12) and (3.13) in the commuting variables $X$ and $Y$, it is noted that Proposition 3.1 corresponds to the identifications

$$
(kp,p') \;=\; (p,kp') \tag{3.26}
$$

of *irreducible* Kac representations, while the analogue of the decompositions (2.4) follow straightforwardly from Proposition 3.2 and the product form of the fusion matrices (3.13). Finally, Corollary 3.3 corresponds to the identifications (2.5). We may thus conclude that, modulo the polynomial $P_{p,p'}(X,Y)$, the matrices defined in (3.12) and (3.13) provide a fusion-matrix realization of the fundamental fusion algebra of $\mathcal{LM}(p,p')$.

Our final objective here is to identify the polynomial ring structure isomorphic to this fusion algebra. First, we argue that $\mathbb{C}[x,y]$ is equivalent to the span of the combinations of Chebyshev polynomials (3.12) and (3.13) used in the realization of the fundamental fusion algebra. Since $U_n(z)$ is a polynomial in $z$ of degree $n$, we have

$$
\mathrm{span}_{\mathbb{C}}\big\{z^n;\ n\in\mathbb{Z}_{0,N}\big\} \;=\; \mathrm{span}_{\mathbb{C}}\big\{U_n(z);\ n\in\mathbb{Z}_{0,N}\big\} \tag{3.27}
$$

Furthermore,

$$
\big(2-\delta_{a,0}\big)T_a(z)U_{pk-1}(z) \;=\; U_{pk-a-1}(z)+U_{pk+a-1}(z) \tag{3.28}
$$

implies that we for $N=\kappa p+\alpha$ where $\kappa\in\mathbb{N}\cup\{0\}$ and $\alpha\in\mathbb{Z}_{0,p-1}$ have

$$
\begin{aligned}
\mathrm{span}_{\mathbb{C}}\big\{z^n;\ n\in\mathbb{Z}_{0,N}\big\} \;=\; \mathrm{span}_{\mathbb{C}}\big\{&\big(2-\delta_{a,0}\big)T_a(z)U_{pk-1}(z),\ \big(2-\delta_{a',0}\big)T_{a'}(z)U_{p\kappa-1}(z);\\
&k\in\mathbb{Z}_{0,\kappa-1},\ a\in\mathbb{Z}_{0,p-1},\ a'\in\mathbb{Z}_{0,\alpha}\big\}
\end{aligned}
\tag{3.29}
$$

For commuting variables $x$ and $y$, we thus have

$$
\begin{aligned}
\mathbb{C}[x,y] \;&=\; \mathrm{span}_{\mathbb{C}}\big\{U_n(x)U_{n'}(y);\ n,n'\in\mathbb{N}\cup\{0\}\big\}\\
&=\; \mathrm{span}_{\mathbb{C}}\big\{\big(2-\delta_{a,0}\big)T_a(x)U_{pk-1}(x)\big(2-\delta_{b,0}\big)T_b(y)U_{p'k'-1}(y);\\
&\qquad k,k'\in\mathbb{N}\cup\{0\},\ a\in\mathbb{Z}_{0,p-1},\ b\in\mathbb{Z}_{0,p'-1}\big\}
\end{aligned}
\tag{3.30}
$$

Now, the matrices $X$ and $Y$ satisfy an infinity of conditions, but as demonstrated above, they are all consequences of a single condition, namely

$$P_{p,p'}(X,Y) \;=\; 0 \tag{3.31}$$

We can therefore conclude that the fusion-matrix realization of the fundamental fusion algebra of the logarithmic minimal model $\mathcal{LM}(p,p')$ is isomorphic to the ring of polynomials in $X$ and $Y$ modulo the ideal defined by (3.31). This is the content of our main result, Proposition 1.1.

# 4 Critical Dense Polymers and Critical Percolation

When choosing a basis in which to examine the fusion matrices associated to the fundamental fusion algebra (2.6), it is natural to separate the set of generators into families. First, there is the finite set of reducible yet indecomposable Kac representations of rank 1. These representations are of the form $(r_0,s_0)$ and there are $(p-1)(p'-1)$ such representations. The remaining infinitely many representations are of the form $\mathcal{R}_{r,s}^{a,b}$ (where $\mathcal{R}_{r,s}^{0,0}=(r,s)$) and are naturally organized into families labeled by the values of $r,s,a,b$ where $r$ and $s$ are given modulo $p$ and $p'$, respectively, cf. (2.6). An example of such a family is thus $\{\mathcal{R}_{pk,s_0}^{p-1,0};\ k\in\mathbb{N}\}$, and every such family is isomorphic to $\mathbb{N}$. By simple inspection of (2.6), it follows that the number of these infinite-dimensional families is

$$N_f \;=\; \frac{(3p-1)(3p'-1)-1}{3} \tag{4.1}$$

This means that the infinite-dimensional fusion matrices are naturally realized as block matrices where each rectangular block is of dimension $(p-1)(p'-1)\times(p-1)(p'-1)$, $(p-1)(p'-1)\times\infty$, $\infty\times(p-1)(p'-1)$ or $\infty\times\infty$, and the total number of blocks is $(N_f+1-\delta_{p,1})^2$. Multiplication or addition of two matching fusion matrices is performed by first treating the blocks as entries of $(N_f+1-\delta_{p,1})\times(N_f+1-\delta_{p,1})$-matrices followed by ordinary multiplication or addition of the matrix blocks as infinite-dimensional matrices. Once the various blocks have been identified, this arithmetic can of course be carried out by introducing a common cut-off to the dimensions of the infinite matrix blocks which is ultimately considered to run off to infinity.

In the two important examples of critical dense polymers $\mathcal{LM}(1,2)$ and critical percolation $\mathcal{LM}(2,3)$, we will now show that the explicitly constructed (infinite-dimensional) fusion matrices $X$ and $Y$ indeed satisfy the conditions underlying our analysis of the polynomial fusion ring above, namely $[X,Y]=0$ and $P_{p,p'}(X,Y)=0$.

## 4.1 Critical Dense Polymers $\mathcal{LM}(1,2)$

In the basis

$$\{(1,1),(2,1),(3,1),\ldots;(1,2),(1,4),(1,6),\ldots;\mathcal{R}_{1,2}^{0,1},\mathcal{R}_{1,4}^{0,1},\mathcal{R}_{1,6}^{0,1},\ldots\} \tag{4.2}$$

we have

$$X \;=\; \begin{pmatrix} D & 0 & 0\\ 0 & D & 0\\ 0 & 0 & D\end{pmatrix},\qquad Y \;=\; \begin{pmatrix} 0 & I & 0\\ 0 & 0 & I\\ 0 & D_+ & 0\end{pmatrix} \tag{4.3}$$

where each of the 9 entries is an infinite-dimensional square matrix with $D$ and $D_+$ defined as

$$D = \begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & 1 & & \\ & 1 & 0 & 1 & \\ & & 1 & 0 & \\ & & & & \ddots \end{pmatrix}, \qquad D_+ = D + 2I \tag{4.4}$$

**Proposition 4.1**

$$[X, Y] = 0 \tag{4.5}$$

and

$$0 = 2P_{1,2}(X, Y) = (X - Y^2 + 2)Y \tag{4.6}$$

**Proof** With every explicitly written matrix entry being an infinite-dimensional matrix, the first identity (4.5) follows from

$$XY = \begin{pmatrix} 0 & D & 0 \\ 0 & 0 & D \\ 0 & D^2 + 2D & 0 \end{pmatrix} = YX \tag{4.7}$$

while the second identity (4.6) follows from

$$X - Y^2 + 2 = \begin{pmatrix} D & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & D \end{pmatrix} - \begin{pmatrix} 0 & 0 & I \\ 0 & D_+ & 0 \\ 0 & 0 & D_+ \end{pmatrix} + \begin{pmatrix} 2I & 0 & 0 \\ 0 & 2I & 0 \\ 0 & 0 & 2I \end{pmatrix} = \begin{pmatrix} D_+ & 0 & -I \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{4.8}$$

and hence

$$(X - Y^2 + 2)Y = \begin{pmatrix} D_+ & 0 & -I \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & I & 0 \\ 0 & 0 & I \\ 0 & D_+ & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{4.9}$$

$\square$

We emphasize that since $Y$ does not have an inverse, it is obvious that, despite the algebraic factorization of $P_{1,2}(X, Y)$, the vanishing condition (4.6) for $P_{1,2}(X, Y)$ is inequivalent to the vanishing condition for the factor $X - Y^2 + 2$. This is clear from (4.8) as well.

In terms of the fundamental fusion matrices $X$ and $Y$, the fusion matrices associated to the representations (4.2) read

$$(k, 1) \leftrightarrow U_{k-1}\Big(\frac{X}{2}\Big), \qquad (1, 2k) \leftrightarrow U_{2k-1}\Big(\frac{Y}{2}\Big), \qquad \mathcal{R}_{1,2k}^{0,1} \leftrightarrow U_{2k-2}\Big(\frac{Y}{2}\Big) + U_{2k}\Big(\frac{Y}{2}\Big) \tag{4.10}$$

where $k \in \mathbb{N}$. According to Proposition 1.1, the fundamental fusion algebra of critical dense polymers $\mathcal{LM}(1, 2)$ is isomorphic to the quotient polynomial ring structure generated by $X$ and $Y$ modulo the ideal defined by (4.6). Abbreviating the ideal by its defining polynomial, we thus have

$$\big\langle (1, 2), (2, 1) \big\rangle_{1,2} \simeq \mathbb{C}[X, Y]/\big(XY - Y^3 + 2Y\big) \tag{4.11}$$

## 4.2 Critical Percolation $\mathcal{LM}(2,3)$

In the basis

$$\begin{aligned}
&\{(1,1),(1,2);(2,1),(4,1),(6,1),\ldots;(2,2),(4,2),(6,2)\ldots;(1,3),(1,6),(1,9),\ldots;\\
&\quad(2,3),(4,3),(6,3),\ldots;\mathcal{R}_{2,1}^{1,0},\mathcal{R}_{4,1}^{1,0},\mathcal{R}_{6,1}^{1,0},\ldots;\mathcal{R}_{2,2}^{1,0},\mathcal{R}_{4,2}^{1,0},\mathcal{R}_{6,2}^{1,0},\ldots;\mathcal{R}_{2,3}^{1,0},\mathcal{R}_{4,3}^{1,0},\mathcal{R}_{6,3}^{1,0},\ldots;\\
&\quad\mathcal{R}_{1,3}^{0,1},\mathcal{R}_{1,6}^{0,1},\mathcal{R}_{1,9}^{0,1},\ldots;\mathcal{R}_{2,3}^{0,1},\mathcal{R}_{2,6}^{0,1},\mathcal{R}_{2,9}^{0,1},\ldots;\mathcal{R}_{1,3}^{0,2},\mathcal{R}_{1,6}^{0,2},\mathcal{R}_{1,9}^{0,2},\ldots;\mathcal{R}_{2,3}^{0,2},\mathcal{R}_{2,6}^{0,2},\mathcal{R}_{2,9}^{0,2},\ldots;\\
&\quad\mathcal{R}_{2,3}^{1,1},\mathcal{R}_{4,3}^{1,1},\mathcal{R}_{6,3}^{1,1},\ldots;\mathcal{R}_{2,3}^{1,2},\mathcal{R}_{4,3}^{1,2},\mathcal{R}_{6,3}^{1,2},\ldots\}
\end{aligned} \tag{4.12}$$

we have

$$X \;=\; \begin{pmatrix}
0 & E_u & E_d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0\\
0 & D_+ & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & D_+ & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & D_+ & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & D_+ & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & D_+ & 0 & 0
\end{pmatrix} \tag{4.13}$$

and

$$Y \;=\; \begin{pmatrix}
\Gamma & 0 & 0 & E_d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & I & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & I & 0 & I & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0\\
0 & 0 & 0 & 2I & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 2I & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0\\
0 & 0 & 0 & D & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & D & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2I & 0 & 0 & 0 & 0 & 0 & I\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & D & 0 & 0 & 0 & 0 & I & 0
\end{pmatrix} \tag{4.14}$$

where

$$\Gamma \;=\; \begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}, \qquad E_u \;=\; \begin{pmatrix} 1 & 0 & 0 & \ldots\\ 0 & 0 & 0 & \ldots\end{pmatrix}, \qquad E_d \;=\; \begin{pmatrix} 0 & 0 & 0 & \ldots\\ 1 & 0 & 0 & \ldots\end{pmatrix} \tag{4.15}$$

Denoting the explicitly written entries of $X$ (and similarly of $Y$) by $X_{i,j}$ where $i,j\in\mathbb{Z}_{1,14}$, the entry $X_{1,1}$ is a $2\times 2$-matrix; the entry $X_{1,j}$ for $j\in\mathbb{Z}_{2,14}$ consists of 2 infinite rows; the entry

$X_{i,1}$ for $i \in \mathbb{Z}_{2,14}$ consists of 2 infinite columns; whereas the entry $X_{i,j}$ for $i,j \in \mathbb{Z}_{2,14}$ is an infinite-dimensional matrix like (4.4).

**Proposition 4.2**

$$[X, Y] \;=\; 0 \tag{4.16}$$

and

$$0 \;=\; 2P_{2,3}(X,Y) \;=\; X\big(X^2 - Y^3 + 3Y - 2\big)\big(Y^2 - 1\big) \tag{4.17}$$

**Proof** As in the case of Proposition 4.1, this proposition follows by direct inspection. □

In terms of the fundamental fusion matrices $X$ and $Y$, the fusion matrices associated to the representations (4.12) read

$$\begin{gathered}
(1,1) \;\leftrightarrow\; 1, \quad (1,2) \;\leftrightarrow\; Y, \quad (2k,1) \;\leftrightarrow\; U_{2k-1}\big(\tfrac{X}{2}\big), \quad (2k,2) \;\leftrightarrow\; U_{2k-1}\big(\tfrac{X}{2}\big)Y \\
(1,3k) \;\leftrightarrow\; U_{3k-1}\big(\tfrac{Y}{2}\big), \quad (2k,3) \;\leftrightarrow\; U_{2k-1}\big(\tfrac{X}{2}\big)\big(Y^2-1\big) \\
\mathcal{R}_{2k,1}^{1,0} \;\leftrightarrow\; XU_{2k-1}\big(\tfrac{X}{2}\big), \quad \mathcal{R}_{2k,2}^{1,0} \;\leftrightarrow\; XU_{2k-1}\big(\tfrac{X}{2}\big)Y, \quad \mathcal{R}_{2k,3}^{1,0} \;\leftrightarrow\; XU_{2k-1}\big(\tfrac{X}{2}\big)\big(Y^2-1\big) \\
\mathcal{R}_{1,3k}^{0,1} \;\leftrightarrow\; YU_{3k-1}\big(\tfrac{Y}{2}\big), \quad \mathcal{R}_{1,3k}^{0,2} \;\leftrightarrow\; \big(Y^2-2\big)U_{3k-1}\big(\tfrac{Y}{2}\big) \\
\mathcal{R}_{2,3k}^{0,1} \;\leftrightarrow\; XYU_{3k-1}\big(\tfrac{Y}{2}\big), \quad \mathcal{R}_{2,3k}^{0,2} \;\leftrightarrow\; X\big(Y^2-2\big)U_{3k-1}\big(\tfrac{Y}{2}\big) \\
\mathcal{R}_{2k,3}^{1,1} \;\leftrightarrow\; XU_{2k-1}\big(\tfrac{X}{2}\big)\big(Y^3-Y\big), \quad \mathcal{R}_{2k,3}^{1,2} \;\leftrightarrow\; XU_{2k-1}\big(\tfrac{X}{2}\big)\big(Y^4-3Y^2+2\big)
\end{gathered} \tag{4.18}$$

where $k \in \mathbb{N}$. According to Proposition 1.1, the fundamental fusion algebra of critical percolation $\mathcal{LM}(2,3)$ is isomorphic to the quotient polynomial ring structure generated by $X$ and $Y$ modulo the ideal defined by (4.17). Abbreviating the ideal by its defining polynomial, we thus have

$$\big\langle(1,2),(2,1)\big\rangle_{2,3} \;\simeq\; \mathbb{C}[X,Y]/\big(X^3Y^2 - X^3 - XY^5 + 4XY^3 - 2XY^2 - 3XY + 2X\big) \tag{4.19}$$

# 5 Conclusion

We have derived a fusion-matrix realization of the fundamental fusion algebra [2, 3] of every logarithmic minimal model $\mathcal{LM}(p,p')$ [1]. The various fusion matrices are all expressed in terms of Chebyshev polynomials in the two infinite-dimensional fundamental fusion matrices $X$ and $Y$ corresponding to the fundamental representations $(2,1)$ and $(1,2)$, respectively. In terms of this realization, we have identified the quotient polynomial ring structure isomorphic to the fundamental fusion algebra itself. This extends the regime of validity of Gepner's result [4] on the existence of such a quotient polynomial ring isomorphic to a rational conformal field theory to the irrational logarithmic minimal models. We have found, though, that the conjectured existence of an associated polynomial fusion potential [4] does not extend to the logarithmic minimal models. We have worked out explicit realizations of the fundamental fusion matrices in the cases of critical dense polymers $\mathcal{LM}(1,2)$ and critical percolation $\mathcal{LM}(2,3)$, and hence of the full fusion-matrix realizations of the associated fundamental fusion algebras. We have verified that these explicit matrices satisfy the basic constraints underlying our construction of the fusion rings.

The fundamental fusion algebras presented in [2, 3] are supported, within a lattice formulation, by extensive numerical studies of associated integrable lattice models. Despite the vastness of this numerical data set, the fusion rules can only be considered conjectural. It is therefore

very reassuring that the fusion algebra is isomorphic to a polynomial fusion ring whose ideal is defined by a single vanishing condition which, in turn, corresponds to the natural identification of the two irreducible highest-weight representations $(2p, p')$ and $(p, 2p')$ of identical conformal weights.


## Acknowledgments

This work is supported by the Australian Research Council. PAP thanks Giuseppe Mussardo and Aldo Delfino for hospitality at the Statistical Physics Section of SISSA and Luc Frappat and Eric Ragoucy for hospitality at LAPP.


# A Chebyshev Polynomials

## A.1 Chebyshev Polynomials of the First Kind

Recursion relation:

$$T_n(x) \;=\; 2xT_{n-1}(x) - T_{n-2}(x), \qquad n = 2, 3, \ldots \tag{A.1}$$

Initial conditions:

$$T_0(x) \;=\; 1, \qquad T_1(x) \;=\; x \tag{A.2}$$

Examples:

$$\begin{aligned} T_2(x) &= 2x^2 - 1 \\ T_3(x) &= 4x^3 - 3x \\ T_4(x) &= 8x^4 - 8x^2 + 1 \\ T_5(x) &= 16x^5 - 20x^3 + 5x \end{aligned} \tag{A.3}$$

## A.2 Chebyshev Polynomials of the Second Kind

Recursion relation:

$$U_n(x) \;=\; 2xU_{n-1}(x) - U_{n-2}(x), \qquad n = 2, 3, \ldots \tag{A.4}$$

Initial conditions:

$$U_0(x) \;=\; 1, \qquad U_1(x) \;=\; 2x \tag{A.5}$$

Examples:

$$\begin{aligned} U_2(x) &= 4x^2 - 1 \\ U_3(x) &= 8x^3 - 4x \\ U_4(x) &= 16x^4 - 12x^2 + 1 \\ U_5(x) &= 32x^5 - 32x^3 + 6x \end{aligned} \tag{A.6}$$

Extension:

$$U_{-1}(x) \;=\; 0 \tag{A.7}$$

Decomposition of product:

$$U_m(x)U_n(x) \;=\; \sum_{j=|m-n|,\,\text{by }2}^{m+n} U_j(x) \tag{A.8}$$

## A.3 Relating Chebyshev Polynomials of the First and Second Kind

Basic relation:

$$2T_n(x) \;=\; U_n(x) - U_{n-2}(x), \qquad n \in \mathbb{N} \tag{A.9}$$

Generalization of the basic relation (A.9):

$$2T_n(x)U_{m-1}(x) \;=\; \begin{cases} U_{n+m-1}(x) - U_{|n-m|-1}(x), & n > m \\ U_{n+m-1}(x), & n = m \\ U_{n+m-1}(x) + U_{|n-m|-1}(x), & n < m \end{cases} \tag{A.10}$$

Applying (A.9) and (A.8) in the given order to the left side of (A.10) yields a difference of two sums which simplifies to the right side of (A.10).

Special expansions, with $p \in \mathbb{N}$:

$$\begin{aligned} U_{(2n+1)p-1}(x) \;&=\; \Big(1 + 2\sum_{j=1}^{n} T_{2jp}(x)\Big)U_{p-1}(x) \\ U_{2np-1}(x) \;&=\; 2\sum_{j=1}^{n} T_{(2j-1)p}(x)U_{p-1}(x) \end{aligned} \tag{A.11}$$

These relations follow by induction in $n$. In particular, the induction step used in establishing the first relation reads

$$\begin{aligned} U_{(2n+1)p-1}(x) \;&=\; 2T_{2np}(x)U_{p-1}(x) + U_{(2n-1)p-1}(x) \\ &=\; 2T_{2np}(x)U_{p-1}(x) + \Big(1 + 2\sum_{j=1}^{n-1} T_{2jp}(x)\Big)U_{p-1}(x) \\ &=\; \Big(1 + 2\sum_{j=1}^{n} T_{2jp}(x)\Big)U_{p-1}(x) \end{aligned} \tag{A.12}$$

where the first equality is a consequence of (A.10). The second relation in (A.11) follows similarly.

Derivative:

$$\partial_x T_n(x) \;=\; nU_{n-1}(x), \qquad n \in \mathbb{N} \cup \{0\} \tag{A.13}$$

## B Fusion Potential

For $1 \leq p < p'$, we now show that the single constraint $M(x,y)=0$, where $M(x,y)=M_{p,p'}(x,y)$ is defined in (3.3) with $[x,y]=0$, cannot be derived from a polynomial potential $V(x,y)$ as the condition defining the local extrema of $V(x,y)$. The conditions for local extrema imply that the partial derivatives of $V(x,y)$ must vanish modulo $M(x,y)$. Also, since $M(x,y)$ must be generated from $V(x,y)$, the former must be expressible as a linear combination of the partial derivatives of the latter. We can thus characterize the polynomial potential $V(x,y)$ by the conditions

$$\begin{aligned} \partial_x V(x,y) &= f(x,y)M(x,y), \qquad \partial_y V(x,y) = g(x,y)M(x,y) \\ \alpha\partial_x V(x,y) + \beta\partial_y V(x,y) &= M(x,y) \end{aligned} \tag{B.1}$$

for some $\alpha, \beta \in \mathbb{C}$ and polynomials $f(x,y)$ and $g(x,y)$. We have four situations, depending on $\alpha$ and $\beta$ being 0 or not, all of which we now discard one by one. Assuming $\alpha=\beta=0$, we are immediately faced with the contradiction $M(x,y)=0$. It is noted that since $M(x,y)$ is asymmetric in its dependence on $x$ and $y$ due to the inequality $p<p'$, the two cases $\alpha=0, \beta\neq 0$ and $\alpha\neq 0, \beta=0$ should be treated separately.

Assuming $\alpha\neq 0, \beta=0$, we integrate $\alpha\partial_x V(x,y)=M(x,y)$ to obtain

$$V(x,y) = \frac{1}{\alpha}\Big(\frac{1}{2p}T_{2p}(x)U_{p'-1}(y) - \frac{1}{p}T_p(x)U_{2p'-1}(y)\Big) + \bar{V}(y) \tag{B.2}$$

for some polynomial $\bar{V}(y)$, thus implying

$$\frac{1}{\alpha}\Big(\frac{1}{2p}T_{2p}(x)U'_{p'-1}(y) - \frac{1}{p}T_p(x)U'_{2p'-1}(y)\Big) + \bar{V}'(y) = g(x,y)\big(U_{2p-1}(x)U_{p'-1}(y) - U_{p-1}(x)U_{2p'-1}(y)\big) \tag{B.3}$$

Considering this as an identification of polynomials in $x$ with focus on the leading terms, we find that

$$\frac{1}{2p\alpha}\big(2^{2p-1}x^{2p}\big)\big(2^{p'-1}(p'-1)y^{p'-2}\big) + \ldots = g(x,y)\big(2^{2p-1}x^{2p-1}\big)\big(2^{p'-1}y^{p'-1}\big) + \ldots \tag{B.4}$$

Matching these for $g(x,y)$ polynomial (in $y$, in particular) requires $p'=1$ and $g(x,y)=0$, but $1 \leq p < p'$.

Assuming $\alpha=0, \beta\neq 0$, we likewise obtain the requirement $p=1$ and $f(x,y)=0$. This implies $\partial_x V(x,y)=0$ and $\partial_y V(x,y) = xU_{p'-1}(y) - U_{2p'-1}(y)$. Integrating the latter with respect to $y$ yields a potential $V(x,y)$ with non-trivial dependence on $x$ in contradiction with $\partial_x V(x,y)=0$.

Assuming $\alpha\neq 0, \beta\neq 0$, polynomial identification yields $\alpha f(x,y)+\beta g(x,y)=1$ and we are left with the two conditions

$$\partial_x V(x,y) = f(x,y)M(x,y), \qquad \partial_y V(x,y) = \frac{1}{\beta}\big(1-\alpha f(x,y)\big)M(x,y) \tag{B.5}$$

We compute the double derivatives

$$\begin{aligned} \partial_y\partial_x V(x,y) &= \partial_y f(x,y)M(x,y) + f(x,y)\partial_y M(x,y) \\ \partial_x\partial_y V(x,y) &= -\frac{\alpha}{\beta}\partial_x f(x,y)M(x,y) + \frac{1}{\beta}\big(1-\alpha f(x,y)\big)\partial_x M(x,y) \end{aligned} \tag{B.6}$$

If $f(x,y)=0$, we have $\partial_y\partial_x V(x,y)=0$ and $\partial_x\partial_y V(x,y)=(1/\beta)\partial_x M(x,y)\neq 0$ so $f(x,y)\neq 0$. From (B.6), we read off the bounds

$$\begin{aligned}\deg_x\big[\partial_y\partial_x V(x,y)\big] &\leq \deg_x f(x,y)+\deg_x M(x,y)\\ \deg_x\big[\partial_x\partial_y V(x,y)\big] &\leq \deg_x f(x,y)+\deg_x M(x,y)-1\end{aligned} \tag{B.7}$$

where $\deg_x h(x,y)$ denotes the degree of $h(x,y)$ as a polynomial in $x$. An inconsistency is thus reached if the first bound is saturated. From

$$\partial_y\partial_x V(x,y) \;=\; U_{2p-1}(x)\partial_y\big[f(x,y)U_{p'-1}(y)\big]-U_{p-1}(x)\partial_y\big[f(x,y)U_{2p'-1}(y)\big] \tag{B.8}$$

and the expansion $f(x,y)=x^{d_f}f_0(y)+\ldots$ where $d_f=\deg_x f(x,y)$ (such that $f_0(y)\neq 0$ since $f(x,y)\neq 0$), we conclude that saturation of the first inequality (B.7) is prevented if and only if $\partial_y\big[f_0(y)U_{p'-1}(y)\big]=0$. Since $p'>1$, the polynomial $U_{p'-1}(y)$ is non-constant implying the sought contradiction $f_0(y)=0$.

Considering (3.18), we thus conclude that the conjectured existence of a polynomial fusion potential in the case of a *rational* conformal field theory [4] does *not* extend to the *irrational* $\mathcal{LM}(p,p')$.

It is noted that having fewer polynomial conditions (here only $M(x,y)=0$) than variables (here $x$ and $y$) is not enough to prevent a polynomial potential from existing. A single polynomial condition given by a function of $\alpha x+\beta y$ only, for example, can be easily integrated to yield the desired potential. It was the particular 'semi-factorized' form of the single condition $M(x,y)=0$ above which allowed us to exclude the possibility of a polynomial potential.

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